Context-Directed Reversals of Signed Permutations

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Abstract

This research examines a generalization of the context-directed reversal operation on signed permutations to an operation on two-rooted graphs. Results include a cdr-sortability condition, characterization of a cdr *and* gcdr *invariant, and graph enumerations.*

Signed Permutations

Signed permutations are arrangements of the first n natural numbers with associated positive and negative signs. Signed permutations are used to model genome recombination.

Pointers are assigned to each side of each entry of a signed permutation, as follows:

$0_{(0,1)}\ [_{(1,2)}2_{(2,3)}\ _{(4,5)}-4_{(3,4)}\ _{(0,1)}1_{(1,2)}\ _{(2,3)}3_{(3,4)}]\ _{(4,5)}5.$

CDR

The cdr (context-directed reversal) operation can be applied to a pointer pair that is associated with entries with different sign. cdr reverses and negates the entries between the pointers.

Example: Applying cdr at $(3,4)$ on the permutation $[2, -4_{(3,4)}, 1, 3_{(3,4)}]$ yields $[2, -4, -3, -1]$.

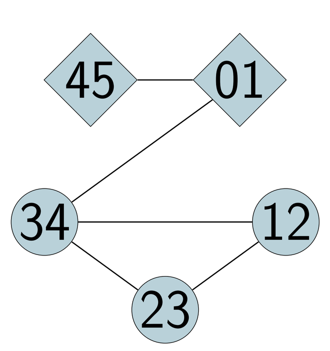
A signed permutation is **cdr-sortable** if some sequence of cdr moves yields the identity permutation $[1, 2, 3, \ldots, n]$.

The Overlap Graph

The **overlap graph** of a signed permutation is construted so that vertices represent pointers and edges represent pointer pairs that follow the pattern $p \ldots q \ldots p \ldots q$.

Example:

$0_{(0,1)}[_{(4,5)}-4_{(3,4)},\ _{(0,1)}1_{(1,2)},\ _{(2,3)}3_{(3,4)},\ _{(1,2)}2_{(2,3)}]_{(4,5)}5$

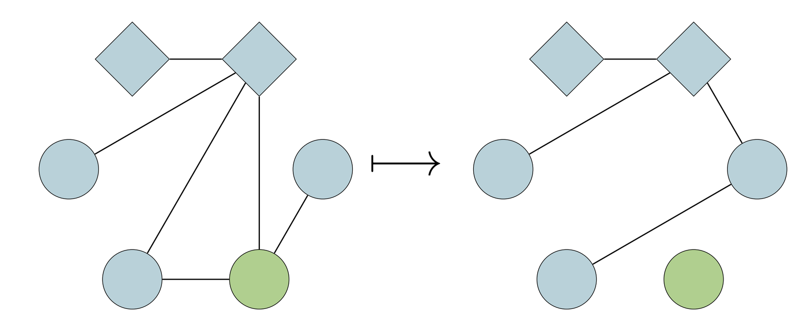


- Each signed permutation generates a unique overlap graph.
- A pointer is **oriented** if it is associated with entries with different signs. A vertex of an overlap graph is oriented if and only if it has odd degree.

Two-Rooted Graphs and GCDR

A **two-rooted graph** is a labeled graph with two distinguished vertices denoted with diamonds, also called **roots**.

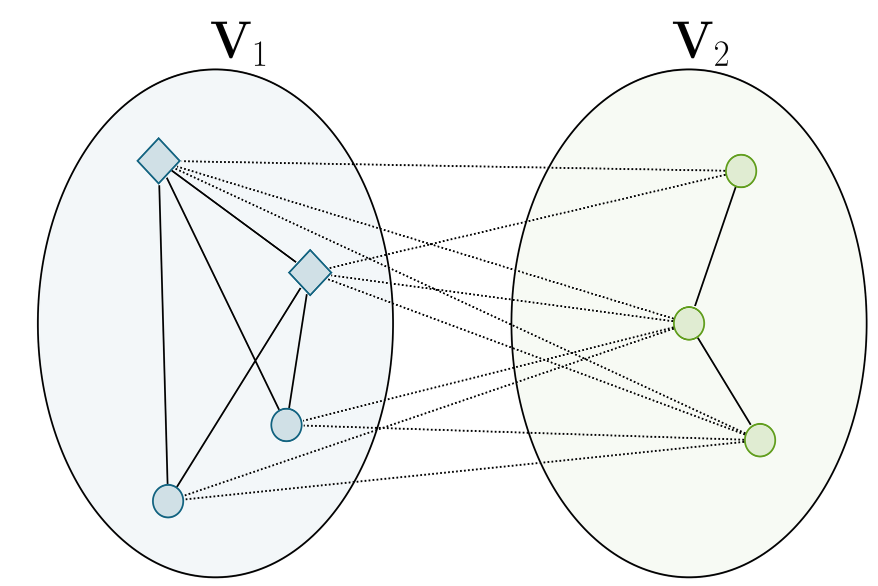
gcdr operates on a non-root oriented vertex v of a two-rooted graph by complementing all of the edges in its neighborhood. For a signed permutation, performing gcdr on a vertex of the overlap graph is equivalent to performing cdr on the corresponding pointer in the permutation.



gcdr applied to the green vertex

Parity Cuts

A **parity cut** of a two-rooted graph $G = (V, E)$ is a partition of V into V_1 and V_2 such that for each non-root $v \in V_1$, v is adjacent to an even number of vertices in V_2, and for each non-root $w \in V_2$, w is adjacent to an even number of vertices in V_1.

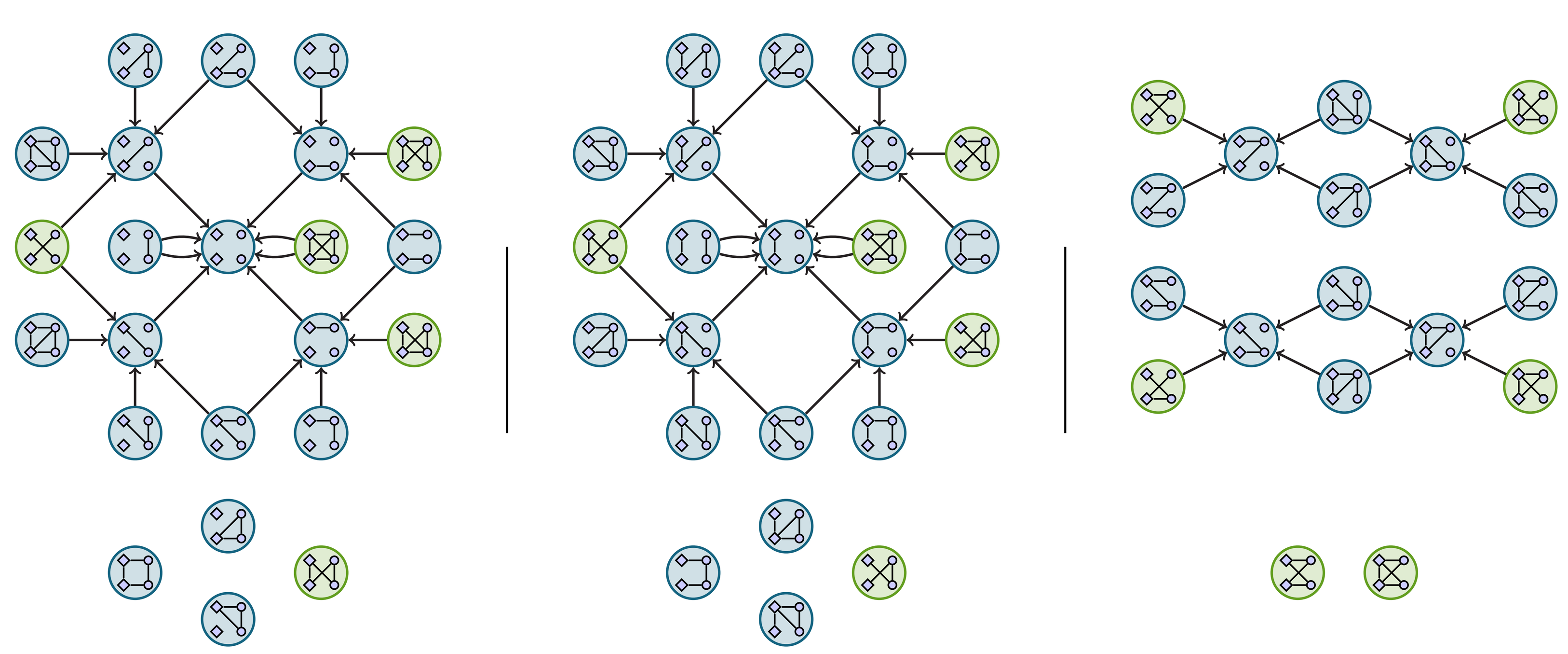


example of a parity cut

The a-b-c Trichotomy

A two-rooted graph G with roots x, y satisfies

- **property (a)** if there is a parity cut $\{V_1, V_2\}$ of G such that $x \in V_1$, $y \in V_2$, x is adjacent to an even number of vertices in V_2, and y is adjacent to an even number of vertices in V_1;
- **property (b)** if there is a parity cut $\{V_1, V_2\}$ of G such that $x \in V_1$, $y \in V_2$, x is adjacent to an odd number of vertices in V_2, and y is adjacent to an odd number of vertices in V_1;
- **property (c)** if there is a parity cut $\{V_1, V_2\}$ of G such that $x, y \in V_1$, and x, y are each adjacent to an odd number of vertices in V_2.



The gcdr-digraph of size $n = 3$: vertices represent graphs of size 3 and edges denote legal gcdr moves between vertices. Blue vertices correspond to signed permutations. This graph forms three nontrivial components for $n \geq 4$ that correspond to (a),(b), and (c). Components (a) and (b) are isomorphic under reverse negative, or complementing the edge between the two roots.

Trichotomy Results

- Every finite two-rooted graph satisfies exactly one of (a), (b), or (c).
- Properties (a),(b), and (c) are invariant under gcdr and cdr.
- Every cdr-sortable permutation satisfies (a).

Enumerative Results

The number of graphs of size n satisfying (a), (b) and (c) is given below for small values of n:

n	$a(n)$	$b(n)$	$c(n)$
1	1	1	0
2	3	3	2
3	23	23	18
4	351	351	322
5	11119	11119	10530
6	703887	703887	689378

For all $n \geq 2$:

$$c(n) = (2^n - 2)a(n-1).$$

This leads to an explicit formula:

$$a(n) = \sum_{k=0}^{n-1} \left(\prod_{i=1}^{k} 2^{i+1} \right) \left(\prod_{i=k+1}^{n-1} (1 - 2^i) \right).$$

Asymptotically,

$$\lim_{n\to\infty} \frac{a(n)}{a(n)+b(n)+c(n)} = \lim_{n\to\infty} \frac{a(n)}{2^{\binom{n+1}{2}}} = \frac{1}{3}.$$

CDR Sortability Theorem

Given a signed permutation π of size n, the following are equivalent:

(i) π is cdr-sortable;

(ii) π satisfies (a), and every component of the verla graph of π contains an oriented vertex.

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Acknowledgments

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